

The rank 1 real Wishart spiked model I. Finite N analysis

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Abstract

This is the first part of a paper that studies the phase transition in the asymptotic limit of the rank 1 real Wishart spiked model. In this paper, we consider N -dimensional real Wishart matrices S in the class $W_{\mathbb{R}}(\Sigma, M)$ in which all but one eigenvalues of Σ is 1. Let the non-trivial eigenvalue of Σ be $1+\tau$, then as $N, M \rightarrow \infty$, with $N/M = \gamma^2$ finite and non-zero, the eigenvalue distribution of S will converge into the Machenko-Pastur distribution inside a bulk region. As τ increases from zero, one starts seeing stray eigenvalues of S outside of the support of the Machenko-Pastur density. As the first of these stray eigenvalues leaves the bulk region, a phase transition will occur in the largest eigenvalue distribution of the Wishart matrix. In this paper will compute the asymptotics of the largest eigenvalue distribution when the phase transition occur. In the this first half of the paper, we will establish the results that are valid for all N and M and will use them to carry out the asymptotic analysis in the second half of the paper, which will follow shortly. In particular, we have derived a formula for the integral $\int_{O(N)} e^{-\text{tr}(XgYg^T)} g^T dg$ when X, Y are symmetric and Y is a rank 1 matrix. This allows us to write down a Fredholm determinant formula for the largest eigenvalue distribution and analyze it using orthogonal polynomial techniques. This approach is very different from a recent paper [10], in which the largest eigenvalue distribution was obtained using stochastic operator method.

1 Introduction

Let X be an $N \times M$ (throughout the paper, we will assume $M > N$ and N is even) matrix such that each column of X is an independent, identical N -variate random variable with normal distribution and zero mean. Let Σ be its covariance matrix, i.e. $\Sigma_{ij} = E(X_{i1}X_{j1})$. Then Σ is an $N \times N$ positive definite symmetric matrix and we denote its eigenvalues by $1+\tau_j$. The matrix S defined by $S = \frac{1}{M}XX^T$ is a real Wishart matrix in the class $W_{\mathbb{R}}(\Sigma, M)$. We can think of each column of X as a draw from a N -variate random variable with the normal distribution and zero mean, then S is the the sample covariance matrix for the samples represented by X . Real Wishart matrices are good models of sample covariance matrices in many situations and have applications in many areas such as finance, genetic studies and climate data. (See [21] for example.)

In many of these applications, one has to deal with data in which both N and M are large, while the ratio N/M is finite and non-zero. In particular, in applications to principle analysis, one would like to study the asymptotic behavior of the largest eigenvalue of S as $N, M \rightarrow \infty$ with $M/N \rightarrow \gamma^2 \geq 1$ fixed.

For many statistical data, it was noted in [21] that in the asymptotic limit, the eigenvalue distribution of the sample covariance matrix will converge to a distribution whose density is given by the Macheenko-Pastur law [23] inside a bulk region (See [4],[5].)

$$\rho(\lambda) = \frac{\gamma}{2\pi\lambda} \sqrt{(\lambda - b_-)(b_+ - \lambda)} \chi_{[b_-, b_+]}, \quad (1.1)$$

where $\chi_{[b_-, b_+]}$ is the characteristic function for the interval $[b_-, b_+]$ and $b_{\pm} = (1 \pm \gamma^{-1})^2$. However, outside of the bulk region, there are often a finite number of large eigenvalues at isolated locations. This behavior prompted the introduction of the spiked model in [21], which are Wishart matrices with a covariance matrix with all but a finite number of eigenvalues that are not equal to one. These non-trivial eigenvalues in the covariance matrix will then be responsible for the spikes that appear in the eigenvalue distribution of the sample covariance matrix. The number of these non-trivial eigenvalues in Σ is called the rank of the spiked model.

Of particular interest is a phase transition that arises in the largest eigenvalue distributions when the first of these spikes starts leaving the bulk region. This phenomenon was first studied in [7] for the complex Wishart spiked model and then in [31] for the rank 1 quaternionic Wishart spiked model. Despite having the most applications, the asymptotics for real Wishart spiked model has not been solved until very recently [10]. The main goal of this paper is to obtain the largest eigenvalue distribution for the rank 1 real Wishart spiked model in the asymptotic limit. In a recent paper [10], the asymptotic largest eigenvalue distribution for the rank 1 real Wishart ensemble was obtained by using a completely different approach to ours. In [10], the authors first use the Householder algorithm to reduce a Wishart matrix into tridiagonal form. Such tridiagonal matrix is then treated as a discrete random Schrödinger operator and by taking an appropriate scaling limit, the authors obtained a continuous random Schrödinger operator on the half-line. By doing so, the authors in [10] bypass the problem of determining the eigenvalue j.p.d.f. for the real Wishart ensemble and obtain the largest eigenvalue distribution in the asymptotic limit.

On the other hand, the approach presented in this paper uses orthogonal polynomial techniques that are closer to those in [7] and [31]. We will now outline our method.

One of the main difficulties in the asymptotic analysis of the real Wishart ensembles is to find a simple expression for the j.p.d.f. of its eigenvalues. Let λ_j be the eigenvalues of the Wishart matrix, then the j.p.d.f. for the real Wishart ensemble is given by

$$P(\lambda) = \frac{1}{Z_{M,N}} |\Delta(\lambda)| \prod_{j=1}^N \lambda_j^{\frac{M-N-1}{2}} \int_{O(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g S g^{-1})} g^T dg, \quad (1.2)$$

where $g^T dG$ is the Haar measure on $O(N)$ and $Z_{M,N}$ is a normalization constant. The expression of the j.p.d.f. for the complex and quarternionic Wishart ensembles are similar. In the complex case, the integral in the j.p.d.f. will be over the unitary group while in the quarternionic case, the integral will be over the symplectic group. One of the main difficulties in the asymptotic analysis of Wishart ensembles is to evaluate the integral in (1.2). In the complex case, this integral can be evaluated using the Harish-Chandra [17] (or Itzykson Zuber [19]) formula, while in the quarternionic case, the integral can be written as an infinite series in terms of Zonal polynomials and such series converges to a simple function in the rank 1 case. For the real case, however, the Harish-Chandra Itzykson Zuber formula does not apply and while the series expression in terms of Zonal polynomials still exists, such series expression do not seem to converge into a simple function. In fact, our first result is that the integral over $O(N)$ in (1.2) is a hyper-elliptic integral in the eigenvalues $\lambda_1, \dots, \lambda_N$.

Theorem 1. *Assuming N is even. Let the non-trivial eigenvalue in the covariance matrix Σ be $1 + \tau$. Then the j.p.d.f. of the eigenvalues in the rank 1 real Wishart spiked model with covariance matrix Σ is given by*

$$P(\lambda) = \tilde{Z}_{M,N}^{-1} \int_{\Gamma} |\Delta(\lambda)| e^{Mt} \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \lambda_j^{\frac{M-N-1}{2}} \left(t - \frac{\tau}{2(1+\tau)} \lambda_j \right)^{-\frac{1}{2}} dt, \quad (1.3)$$

where Γ is a contour that encloses all the points $\frac{\tau}{2(1+\tau)}\lambda_1, \dots, \frac{\tau}{2(1+\tau)}\lambda_N$ that is oriented in the counter-clockwise direction and $\tilde{Z}_{M,N}$ is the normalization constant. The branch cuts of the square root $\left(t - \frac{\tau}{2(\tau+1)}x \right)^{-\frac{1}{2}}$ is chosen to be the line $\arg(t - \frac{\tau}{2(\tau+1)}x) = \pi$.

We will present two different proofs of this in the paper. The first one is a geometric proof which involves choosing a suitable set of coordinates on $O(N)$ and decompose the Haar measure into two parts so that the integral in (1.2) can be evaluated. This will be achieved in Sections 2 and 3. The second proof is an algebraic proof that uses the Zonal polynomial expansion to verify the formula in Theorem 1. This proof will be given in the Appendix where integral formulae of the form (1.3) for the complex and quarternionic Wishart ensembles will also be derived.

Remark 1. *The integral formula derived here is very similar to a more general formula in [8], in which the matrix integral over $O(N)$ is given by*

$$\int_{O(N)} e^{-\text{tr}(XgYg^{-1})} g^T dg \propto \int \frac{e^{\text{tr}(S)}}{\prod_{j=1}^N \det(S - y_j X)} dS$$

where the integral of S is over $\sqrt{-1}$ times the space of $N \times N$ real symmetric matrices and y_j are the eigenvalues of Y . The measure dS is the flat Lebesgue measure on this space.

From the expression of the j.p.d.f., we see that the largest eigenvalue distribution is given by

$$\begin{aligned}\mathbb{P}(\lambda_{max} < z) &= \int_{\lambda_1 \leq \dots \leq \lambda_N \leq z} \dots \int P(\lambda) d\lambda_1 \dots d\lambda_N, \\ &= \tilde{Z}_{M,N}^{-1} \int_{\Gamma} e^{Mt} \int_{\lambda_1 \leq \dots \leq \lambda_N \leq z} \dots \int |\Delta(\lambda)| \prod_{j=1}^N w(\lambda_j) d\lambda_1 \dots d\lambda_N dt\end{aligned}\tag{1.4}$$

where $w(x)$ is

$$w(x) = e^{-\frac{M}{2}x} x^{\frac{M-N-1}{2}} \left(t - \frac{\tau}{2(1+\tau)}x \right)^{-\frac{1}{2}}\tag{1.5}$$

and Γ is chosen such that it intersects $(0, \infty)$ encloses the interval $[0, z]$.

We can analyze the integrand as in [28], [29] and [30]. By an identity of Bruijn [11], we can express the multiple integral as a Pfaffian.

$$\begin{aligned}&\int_{\lambda_1 \leq \dots \leq \lambda_N \leq y} \dots \int |\Delta(\lambda)| \prod_{j=1}^N w(\lambda_j) d\lambda_1 \dots d\lambda_N \\ &= Pf \left(\langle (1 - \chi_{[z, \infty)})r_j(x), (1 - \chi_{[z, \infty)})r_k(y) \rangle_1 \right).\end{aligned}\tag{1.6}$$

where $r_j(x)$ is an arbitrary sequence of degree j monic polynomials and $\langle f, g \rangle_1$ is the skew product

$$\langle f, g \rangle_1 = \int_0^\infty \int_0^\infty \epsilon(x-y) f(x) g(y) w(x) w(y) dx dy.\tag{1.7}$$

where $\epsilon(x) = \frac{1}{2} \text{sgn}(x)$. In defining the skew product, the contour of integration will be defined such that if t is too close to $(0, \infty)$, then the interval $(0, \infty)$ will be deformed appropriately into the upper or lower half plane such that the integral is well defined. Such deformation will not affect the value of the Pfaffian as Γ will not intersect the integration paths on the left hand side of (1.6). Then by following the method in [28], [29] and [30], we can write the Pfaffian as the square root of a Fredholm determinant. Let \mathcal{M} be the moment matrix with entries $\langle r_j, r_k \rangle_1$, then we have

$$Pf \left(\langle (1 - \chi_{[z, \infty)})(x)r_j(x), (1 - \chi_{[z, \infty)})(y)r_k(y) \rangle_1 \right) = \sqrt{\det \mathcal{M}(t)} \sqrt{\det (I - K\chi_{[z, \infty)}),}$$

where K is the operator whose kernel is given by

$$K(x, y) = \begin{pmatrix} S_1(x, y) & -\frac{\partial}{\partial y} S_1(x, y) \\ IS_1(x, y) & S_1(y, x) \end{pmatrix}\tag{1.8}$$

and $S_1(x, y)$ and $IS_1(x, y)$ are the kernels

$$\begin{aligned}S_1(x, y) &= - \sum_{j,k=0}^{N-1} r_j(x) w(x) \mu_{jk} \epsilon(r_k w)(y), \\ IS_1(x, y) &= - \sum_{j,k=0}^{N-1} \epsilon(r_j w)(x) \mu_{jk} \epsilon(r_k w)(y)\end{aligned}\tag{1.9}$$

and μ_{jk} is the inverse of the matrix \mathcal{M} . As shown in [32], the kernel can now be expressed in terms of the Christoffel Darboux kernel of some suitable orthogonal polynomials, together with a correction term which gives rise to a finite rank perturbation to the Christoffel Darboux kernel. In this paper, we introduce a new proof of this using skew orthogonal polynomials and their representations as multi-orthogonal polynomials. By using ideas from [1] to write skew orthogonal polynomials in terms of orthogonal polynomials, we can express the skew orthogonal polynomials with respect to the weight $w(x)$ in terms of a sum of Laguerre polynomials. Let $\pi_{k,1}$ be the monic skew orthogonal polynomials with respect to the weight $w(x)$.

$$\langle \pi_{2k+1,1}, y^j \rangle_1 = \langle \pi_{2k,1}, y^j \rangle_1 = 0, \quad j = 0, \dots, 2k-1. \quad (1.10)$$

Then we can write these down in terms of Laguerre polynomials.

Proposition 1. *Let L_k be the monic Laguerre polynomials respect to the weight $w_0(x)$*

$$\int_0^\infty L_k(x) L_j(x) w_0(x) dx = \delta_{jk} h_{j,0}, \quad w_0(x) = x^{M-N} e^{-Mx}.$$

If $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$, then the skew orthogonal polynomials $\pi_{2k,1}$ and $\pi_{2k+1,1}$ both exist and $\pi_{2k,1}$ is unique while $\pi_{2k+1,1}$ is unique up to an addition of a multiple of $\pi_{2k,1}$. Moreover, we have $\langle L_{2k}, L_{2k-1} \rangle_1 = 0$ and the skew orthogonal polynomials are given by

$$\begin{aligned} \pi_{2k,1} &= L_{2k} - \frac{\langle L_{2k}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1}, \\ \pi_{2k+1,1} &= L_{2k+1} - \frac{\langle L_{2k+1}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1} + \frac{\langle L_{2k+1}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-2} + c\pi_{2k,1}, \end{aligned}$$

where c is an arbitrary constant.

Next, by representing skew orthogonal polynomials as multi-orthogonal polynomials and write them in terms of the solution of a Riemann-Hilbert problem as in [26], we can apply the results of [15] and [6] to express the kernel $S_1(x, y)$ as a finite rank perturbation of the Christoffel Darboux kernel of the Laguerre polynomials.

Theorem 2. *Let $S_1(x, y)$ defined by (1.9) and choose the sequence of monic polynomials $r_j(x)$ such that $r_j(x)$ are arbitrary degree j monic polynomials that are independent on t and $r_j(x) = \pi_{j,1}(x)$ for $j = N-2, N-1$. Then we have*

$$\begin{aligned} S_1(x, y) - K_2(x, y) &= \\ \epsilon \begin{pmatrix} \pi_{N+1,1} w & \pi_{N,1} w \end{pmatrix} (y) \begin{pmatrix} 0 & -\frac{M\tilde{\tau}}{2h_{N-1,0}} \\ -\frac{M\tilde{\tau}}{2h_{N-2,0}} & \frac{Mt-\tilde{\tau}(M+N)}{2h_{N-1,0}} \end{pmatrix} \begin{pmatrix} L_{2N-2}(x) \\ L_{2N-1}(x) \end{pmatrix} w(x) \end{aligned} \quad (1.11)$$

where $K_2(x, y)$ is the kernel of the Laguerre polynomials

$$K_2(x, y) = \left(\frac{y(t - \tilde{\tau}y)}{x(t - \tilde{\tau}x)} \right)^{\frac{1}{2}} w_0^{\frac{1}{2}}(x) w_0^{\frac{1}{2}}(y) \frac{L_N(x) L_{N-1}(y) - L_N(y) L_{N-1}(x)}{h_{N-1,0}(x - y)}$$

Note that the correction term on the right hand side of (1.11) is the kernel of a finite rank operator. Its asymptotics can be computed using the known asymptotics of the Laguerre polynomials and the method in [12] and [14]. The actual asymptotic analysis of this correction term, however, is particularly tedious as one would need to compute the asymptotics of the skew orthogonal polynomials up to the third leading order term due to cancelations. To compute the contribution from the determinant $\det \mathcal{M}$, we derive the following expression for the logarithmic derivative of $\det \mathcal{M}$.

Proposition 2. *Let \mathcal{M} be the moment matrix with entries $\langle r_j, r_k \rangle_1$, where the sequence of monic polynomials $r_j(x)$ is chosen such that $r_j(x)$ are arbitrary degree j monic polynomials that are independent on t and $r_j(x) = \pi_{j,1}(x)$ for $j = N - 2, N - 1$. Then the logarithmic derivative of $\det \mathcal{M}$ with respect to t is given by*

$$\frac{\partial}{\partial t} \log \det \mathcal{M} = \int_{\mathbb{R}_+} \frac{S_1(x, x)}{t - \tilde{\tau}x} dx, \quad (1.12)$$

This then allows us to express the largest eigenvalue distribution $\mathbb{P}(\lambda_{max} < z)$ as an integral of Fredholm determinant.

Theorem 3. *The largest eigenvalue distribution of the rank 1 real Wishart ensemble can be written in the following integral form.*

$$\mathbb{P}(\lambda_{max} < z) = C \int_{\Gamma} \exp \left(Mt + \int_{c_0}^t \int_{\mathbb{R}_+} \frac{S_1(x, x)}{s - \tilde{\tau}x} dx ds \right) \sqrt{\det(I - K\chi_{[z, \infty)})} dt. \quad (1.13)$$

for some constant c_0 and K is the operator with kernel given by (1.9). The integration contour Γ is a close contour that encloses the interval $[0, z]$ in the anti-clockwise direction.

In the asymptotic limit, we will be able to evaluate the t integral in (1.13) using steepest descent analysis. We shall see that the phase transition occurs when the saddle point in t is such that the singularity $t/\tilde{\tau}$ in the weight $w(x)$ lies within a distance of order $N^{-\frac{2}{3}}$ to the end point b_+ in (1.1). In this case, the factor $(t - \tilde{\tau}x)^{-\frac{1}{2}}$ in the weight $w(x)$ will significantly alter the behavior of the correction term in (1.11) and gives us a phase transition in the largest eigenvalue distribution.

In this first part of the paper, we shall carry out the analysis when N and M are finite and establish the results that are needed in the asymptotic analysis. Throughout the paper, we shall assume that N is even and that $M - N > 0$.

2 Haar measure on $SO(N)$

In this section, we will find a convenient set of coordinate on $O(N)$ to evaluate the integral

$$\int_{O(N)} e^{-\frac{M}{2} \text{tr}(\Sigma^{-1} g S g^{-1})} g^T dg$$

that appears in the expression of the j.p.d.f. (1.2). As both Σ^{-1} and S are symmetric matrices, they can be diagonalized by matrices in $O(N)$. We can therefore replace both Σ^{-1} and S by the diagonal matrices Σ_d^{-1} and Λ_d .

$$\Sigma_d^{-1} = \text{diag} \left(\frac{1}{1 + \tau_1}, \dots, \frac{1}{1 + \tau_N} \right),$$

$$\Lambda_d = \text{diag} (\lambda_1, \dots, \lambda_N)$$

The group $O(N)$ has two connected components, $SO(N)$ and $O_-(N)$ that consists of orthogonal matrices that have determinant 1 and -1 respectively. Let T be the matrix

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{N-2} \end{pmatrix},$$

then the left multiplication by T defines an diffeomorphism from $O_-(N)$ to $SO(N)$. In particular, we can write the integral over $O(N)$ in (1.2) as

$$\begin{aligned} I(\Sigma, \Lambda) &= \int_{O(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} g \Lambda_d g^{-1})} g^T dg, \\ &= \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} g \Lambda_d g^{-1})} g^T dg + \int_{O_-(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} g \Lambda_d g^{-1})} g^T dg \\ &= \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} g \Lambda_d g^{-1})} g^T dg + \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} T g \Lambda_d g^{-1} T^{-1})} g^T dg \\ &= \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} g \Lambda_d g^{-1})} g^T dg + \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\tilde{\Sigma}_d^{-1} g \Lambda_d g^{-1})} g^T dg, \end{aligned}$$

where $\tilde{\Sigma}_d$ is the diagonal matrix with the first two entries of Σ_d swapped.

$$\tilde{\Sigma}_d^{-1} = \text{diag} \left(\frac{1}{1 + \tau_2}, \frac{1}{1 + \tau_1}, \dots, \frac{1}{1 + \tau_N} \right).$$

Note that $g^T dg$ is also the Haar measure on $SO(N)$.

As we are considering the rank 1 spiked model, we let $\tau_1 = \dots = \tau_{N-1} = 0$ and $\tau_N = \tau$. Therefore $\tilde{\Sigma}_d = \Sigma_d$ and we have

$$I(\Sigma, \Lambda) = 2 \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} g \Lambda_d g^{-1})} g^T dg \quad (2.1)$$

Let g_{ij} be the entries of $g \in SO(N)$. Then the integral I can be written as

$$\begin{aligned} I(\Sigma, \Lambda) &= 2 \int_{SO(N)} e^{-\frac{M}{2} \text{tr}(\Sigma_d^{-1} g \Lambda_d g^{-1})} g^T dg, \\ &= 2 \int_{SO(N)} e^{-\frac{M}{2} \text{tr}((\Sigma_d^{-1} - I_N) g \Lambda_d g^{-1})} e^{-\frac{M}{2} \text{tr}(g \Lambda_d g^{-1})} g^T dg, \\ &= 2 \prod_{j=1}^N e^{-\frac{M}{2} \lambda_j} \int_{SO(N)} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j g_{jN}^2} g^T dg, \end{aligned}$$

We will now find an expression of the Haar measure that allows us to compute the integral $I(\Sigma, \Lambda)$.

First we will define a set of coordinates on $SO(N)$ that is convenient for our purpose. We will then express the Haar measure on $SO(N)$ in terms of these coordinates.

An element $g \in SO(n)$ can be written in the following form

$$g = (\vec{g}_1, \dots, \vec{g}_n), \quad |\vec{g}_i| = 1, \quad \vec{g}_i \cdot \vec{g}_j = \delta_{ij}, \quad i, j = 1, \dots, n.$$

This represents $SO(N)$ as the set of orthonormal frames in \mathbb{R}^N with positive orientation whose coordinate axis are given by the vectors \vec{g}_i . As the vector \vec{g}_N is a unit vector, we can write its components as

$$\begin{aligned} g_{1N} &= \cos \phi_1, & g_{jN} &= \prod_{k=1}^{j-1} \sin \phi_k \cos \phi_j, & j &= 2, \dots, n-1, \\ g_{NN} &= \prod_{k=1}^{N-1} \sin \phi_k \end{aligned} \tag{2.2}$$

The remaining vectors $\vec{g}_1, \dots, \vec{g}_{N-1}$ form an orthonormal frame with positive orientation in a copy of \mathbb{R}^{N-1} that is orthogonal to \vec{g}_N . Therefore the set of vectors $\vec{g}_1, \dots, \vec{g}_{N-1}$ can be identified with $SO(N-1)$. To be precise, let \vec{u} be a unit vector in \mathbb{R}^N and let $G(\vec{u}) \in SO(N)$ be a matrix that maps \vec{u} to the vector $(0, \dots, 0, 1)^T$. Then since G is orthogonal, we have

$$G(\vec{g}_N)\vec{g}_j = (v_{j1}, \dots, v_{j,N-1}, 0)^T, \quad j < N \tag{2.3}$$

In particular, the matrix V whose entries are given by v_{ij} for $1 \leq i, j \leq N-1$ is in $SO(N-1)$. A set of coordinates on $SO(N)$ can therefore be given by

$$g = (\vec{g}_N, V). \tag{2.4}$$

In the above equation, \vec{g}_N is identified with the coordinates ϕ_j in (2.2), while the matrix V identify with the coordinates on $SO(N-1)$ that correspond to V . In terms of these coordinates, the left action of an element $S \in SO(N)$ on g is given by the following.

$$\begin{aligned} Sg &= (S\vec{g}_1, \dots, S\vec{g}_{N-1}, S\vec{g}_N)^T \\ &= (SG(\vec{g}_N)^{-1}\vec{v}_1, \dots, SG(\vec{g}_N)^{-1}\vec{v}_{N-1}, S\vec{g}_N)^T \end{aligned}$$

Then as in (2.3), we have

$$G(S\vec{g}_N)SG(\vec{g}_N)^{-1}\vec{v}_j = (\tilde{v}_{j1}, \dots, \tilde{v}_{j,N-1}, 0)^T.$$

The matrix \tilde{V} whose entries are given by \tilde{v}_{ij} are again in $SO(N-1)$, therefore the matrix $G(S\vec{g}_N)SG(\vec{g}_N)^{-1}$ is of the form

$$G(S\vec{g}_N)SG(\vec{g}_N)^{-1} = \begin{pmatrix} \tilde{S}_{N-1} & \vec{s} \\ 0 & s_N \end{pmatrix} \tag{2.5}$$

From the fact that $G(S\vec{g}_N)SG(\vec{g}_N)^{-1}$ is an orthogonal matrix, it is easy to check that $\vec{s} = 0$ and $s_N = \pm 1$. To determine s_N , let us consider the action of $G(S\vec{g}_N)SG(\vec{g}_N)^{-1}$ on $(0, 0, \dots, 1)^T$. We have

$$G(S\vec{g}_N)SG(\vec{g}_N)^{-1}(0, 0, \dots, 1)^T = G(S\vec{g}_N)S\vec{g}_N = (0, 0, \dots, 1)^T$$

Therefore $s_N = 1$ and \tilde{S}_{N-1} is in $SO(N-1)$. The action of S on g is therefore given by

$$Sg = \left(S\vec{g}_N, \tilde{S}_{N-1}V \right). \quad (2.6)$$

We will now write the Haar measure on $SO(N)$ in terms the coordinates (2.4). These coordinates give a local diffeomorphism between $SO(N)$ and $S^{N-1} \times SO(N-1)$ as $\vec{g}_N \in S^{N-1}$ and $V \in SO(N-1)$. Let dX be a measure on S^{N-1} that is invariant under the action of $SO(N)$ and $V^T dV$ be the Haar measure on $SO(N-1)$, then the following measure

$$dH = dX \wedge V^T dV,$$

is invariant under the left action of $SO(N)$. Let $S \in SO(N)$, then its action on the point (\vec{g}_N, V) is given by (2.6), where \tilde{S}_{N-1} depends only on the coordinates $\phi_1, \dots, \phi_{N-1}$. Therefore under the action of S , the measure dH becomes

$$dH \rightarrow dX \wedge V^T \tilde{S}_{N-1}^T \tilde{S}_{N-1} dV = dX \wedge V^T dV, \quad (2.7)$$

as dX is invariant under the action of S . Therefore if we can find a measure on S^{N-1} that is invariant under the action of $SO(N)$, then $dX \wedge V^T dV$ will give us a left invariant measure on $SO(N)$. Since the left invariant measure on a compact group is also right invariant, this will give us the Haar measure on $SO(N)$. As the metric on S^{N-1} is invariant under the action of $SO(N)$, it is clear that the volume form on S^{N-1} is invariant under the action of $SO(N)$. Let dX be the volume form on S^{N-1} , then from (2.7), we see that the measure $dX \wedge V^T dV$ is invariant under the action of $SO(N)$.

Proposition 3. *Let dX be the volume form on S^{N-1} given by*

$$dX = \sin^{N-2}(\phi_1) \sin^{N-1}(\phi_2) \dots \sin(\phi_{N-2}) \wedge_{j=1}^{N-1} d\phi_j$$

in terms of the coordinates $\phi_1, \dots, \phi_{N-1}$ in (2.2) and (2.4), then the Haar measure on $SO(N)$ is equal to a constant multiple of

$$dH = dX \wedge V^T dV,$$

where $V^T dV$ is the Haar measure on $SO(N-1)$ in terms of the coordinates (2.4).

We can now compute the integral $I(\Sigma, \Lambda)$.

3 An integral formula for the j.p.d.f.

By using the expression of the Haar measure derived in the last section, we can now write the integral $I(\Sigma, \Lambda)$ as

$$\begin{aligned} I(\Sigma, \Lambda) &= 2 \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \int_{SO(N)} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j g_{jN}^2} g^T dg, \\ &= 2 \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \int_{SO(N-1)} V^T dV \int_{S^{N-1}} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j g_{jN}^2} dX, \\ &= 2C \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \int_{S^{N-1}} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j g_{jN}^2} dX, \end{aligned}$$

for some constant C , where the $N-1$ sphere S^{N-1} in the above formula is defined by $\sum_{j=1}^N g_{jN}^2 = 1$ and dX is the volume form on it. If we let $g_{jN} = x_j$, then the above can be written as

$$I(\Sigma, \Lambda) = 2C \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \int_{\mathbb{R}^N} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j x_j^2} \delta\left(\sum_{j=1}^N x_j^2 - 1\right) dx_1 \dots dx_N. \quad (3.1)$$

This can be seen most easily by the use of polar coordinates in \mathbb{R}^N , which are given by

$$\begin{aligned} x_1 &= r \cos \phi_1, \quad x_j = r \prod_{k=1}^{j-1} \sin \phi_k \cos \phi_j, \quad j = 2, \dots, N-1, \\ x_N &= r \prod_{k=1}^{N-1} \sin \phi_k, \end{aligned}$$

Then the volume form in \mathbb{R}^N is given by

$$dx_1 \dots dx_N = r^{N-1} \sin^{N-2} \phi_1 \dots \sin \phi_{N-2} dr d\phi_1 \dots d\phi_{N-1}$$

Therefore in terms of polar coordinates, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j x_j^2} \delta\left(\sum_{j=1}^N x_j^2 - 1\right) dx_1 \dots dx_N \\ &= \int_0^\pi d\phi_1 \int_0^{2\pi} d\phi_2 \dots \int_0^{2\pi} d\phi_{N-1} \int_0^\infty dr \delta\left(\sum_{j=1}^N r^2 - 1\right) r^{N-1} \\ &\times e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j x_j^2} \sin^{N-2} \phi_1 \dots \sin \phi_{N-2} \\ &= \int_{S^{N-1}} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j x_j^2} dX. \end{aligned}$$

To compute the integral $I(\Sigma, \Lambda)$, we use a method in the studies of random pure quantum systems [24]. The idea is to consider the Laplace transform of the function $I(\Sigma, \Lambda, t)$ defined by

$$I(\Sigma, \Lambda, t) = 2C \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \int_{\mathbb{R}^N} e^{\frac{\tau M}{2(1+\tau)} \sum_{j=1}^N \lambda_j x_j^2} \delta\left(\sum_{j=1}^N x_j^2 - t\right) dx_1 \dots dx_N,$$

then $I(\Sigma, \Lambda, 1) = I(\Sigma, \Lambda)$. The Laplace transform of $I(\Sigma, \Lambda, t)$ in the variable t is given by

$$\int_0^\infty e^{-st} I(\Sigma, \Lambda, t) dt = 2C \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \int_{\mathbb{R}^N} e^{\sum_{j=1}^N (-s + \frac{\tau M}{2(1+\tau)} \lambda_j) x_j^2} dx_1 \dots dx_N$$

Then, provided $\text{Re}(s) > \max_j(\lambda_j)$, the integral can be computed explicitly to obtain

$$\int_0^\infty e^{-st} I(\Sigma, \Lambda, t) dt = 2C \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \left(s - \frac{\tau M}{2(1+\tau)} \lambda_j\right)^{-\frac{1}{2}}$$

Taking the inverse Laplace transform, we obtain an integral expression for $I(\Sigma, \Lambda)$.

$$I(\Sigma, \Lambda) = \frac{C}{\pi i} \int_{\Gamma} e^s \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \left(s - \frac{\tau M}{2(1+\tau)} \lambda_j\right)^{-\frac{1}{2}} ds,$$

where Γ is a contour that encloses all the points $\frac{\tau M}{2(1+\tau)} \lambda_1, \dots, \frac{\tau M}{2(1+\tau)} \lambda_N$ that is oriented in the counter-clockwise direction. Rescaling the variable s to $s = Mt$, we obtain

$$I(\Sigma, \Lambda) = \frac{M^{1-\frac{N}{2}} C}{\pi i} \int_{\Gamma} e^{Mt} \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \left(t - \frac{\tau}{2(1+\tau)} \lambda_j\right)^{-\frac{1}{2}} dt,$$

This then give us an integral expression for the j.p.d.f.

Theorem 4. *Let the non-trivial eigenvalue in the covariance matrix Σ be $1 + \tau$. Then the j.p.d.f. of the eigenvalues in the rank 1 real Wishart spiked model with covariance matrix Σ is given by*

$$P(\lambda) = \tilde{Z}_{M,N}^{-1} \int_{\Gamma} |\Delta(\lambda)| e^{Mt} \prod_{j=1}^N e^{-\frac{M}{2}\lambda_j} \lambda_j^{\frac{M-N-1}{2}} \left(t - \frac{\tau}{2(1+\tau)} \lambda_j\right)^{-\frac{1}{2}} dt, \quad (3.2)$$

where Γ is a contour that encloses all the points $\frac{\tau}{2(1+\tau)} \lambda_1, \dots, \frac{\tau}{2(1+\tau)} \lambda_N$ that is oriented in the counter-clockwise direction and $\tilde{Z}_{M,N}$ is the normalization constant. The branch cuts of the square root $\left(a - \frac{\tau}{2(\tau+1)} x\right)^{-\frac{1}{2}}$ are chosen to be the line $\arg(a - \frac{\tau}{2(\tau+1)} x) = \pi$.

For the purpose of computing the largest eigenvalue distribution $\mathbb{P}(\lambda_{\max} \leq z)$, we can assume that the eigenvalues are all smaller than or equal to a constant z .

4 Skew orthogonal polynomials

As explain in the introduction, we need to find the skew orthogonal polynomials with the weight (1.5). Let us denote $\frac{\tau}{2(\tau+1)}$ by $\tilde{\tau}$ and consider the skew orthogonal polynomials with respect to the weight

$$w(x) = e^{-\frac{Mx}{2}} x^{\frac{M-N-1}{2}} (t - \tilde{\tau}x)^{-\frac{1}{2}}. \quad (4.1)$$

We shall use the ideas in [1] to express the skew orthogonal polynomials in terms of a linear combinations of Laguerre polynomials.

Let $H_j(x)$ to be the degree $j + 2$ polynomial

$$H_j(x) = \frac{d}{dx} (x^{j+1} (t - \tilde{\tau}x) w(x)) w^{-1}(x), \quad j \geq 0.$$

Then as we assume $M - N > 0$, it is easy to see that

$$\langle f(x), H_j(y) \rangle_1 = \langle f(x), x^j \rangle_2, \quad (4.2)$$

for any $f(x)$ such that $\int_0^\infty f(x) w(x) dx$ is finite, where the product $\langle \rangle_2$ is defined by

$$\langle f(x) g(x) \rangle_2 = \int_0^\infty f(x) g(x) w_0(x) dx, \quad w_0(x) = x^{M-N} e^{-Mx}. \quad (4.3)$$

Note that $w_0(x)$ is not the square of $w(x)$. The fact that $w_0(x)$ is the weight for the Laguerre polynomials allows us to express the skew orthogonal polynomials for the weight (4.1) in terms of Laguerre polynomials.

In particular, this implies that the conditions (1.10) is equivalent to the following conditions

$$\begin{aligned} \langle \pi_{2k,1}, y^j \rangle_1 &= 0, \quad j = 0, 1, \\ \langle \pi_{2k,1}, y^j \rangle_2 &= 0, \quad j = 0, \dots, 2k - 3. \end{aligned} \quad (4.4)$$

and the exactly same conditions for $\pi_{2k+1,1}(x)$. In particular, the second condition implies the skew orthogonal polynomials can be written as

$$\begin{aligned} \pi_{2k,1}(x) &= L_{2k}(x) + \gamma_{2k,1} L_{2k-1}(x) + \gamma_{2k,2} L_{2k-2}(x), \\ \pi_{2k+1,1}(x) &= L_{2k+1}(x) + \gamma_{2k+1,0} L_{2k}(x) + \gamma_{2k+1,1} L_{2k-1}(x) + \gamma_{2k+1,2} L_{2k-2}(x), \end{aligned}$$

where $L_j(x)$ are the degree j monic Laguerre polynomials that are orthogonal with respect to the weight $w_0(x)$.

$$\begin{aligned} L_n(x) &= \frac{(-1)^n e^{Mx} x^{-M+N}}{M^n} \frac{d^n}{dx^n} (e^{-Mx} x^{n+M-N}), \\ &= x^n - \frac{(M - N + n)n}{M} x^{n-1} + O(x^{n-2}). \end{aligned} \quad (4.5)$$

The constants $\gamma_{k,j}$ are to be determined from the first condition in (4.4). We will now show that if $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$, then the skew orthogonal polynomials $\pi_{2k,1}$ and $\pi_{2k+1,1}$ exist and that $\pi_{2k,1}$ is unique. First let us show that the first condition in (4.4) is equivalent to

$$\langle \pi_{2k,1}, L_{2k-j} \rangle_1 = 0, \quad j = 1, 2.$$

To do this, we will first define a map ϱ_N from the span of L_{2k-1} and L_{2k-2} to the span of y and 1.

Let $P(x)$ be a polynomial of degree m . Then we can write the polynomial $P(x)$ as

$$P(x) = \frac{d}{dx} (q(x)x(t - \tilde{\tau}x)w(x)) w^{-1}(x) + R(x) \quad (4.6)$$

where $q(x)$ is a polynomial of degree $m - 2$ and $R(x)$ is a polynomial of degree less than or equal to 1. By writing down the system of linear equations satisfied by the coefficients of $q(x)$ and $R(x)$, we see that the polynomials $q(x)$ and $R(x)$ are uniquely defined for any given $P(x)$. In particular, the map $f : P(x) \mapsto R(x)$ is a well-defined linear map from the space of polynomial to the space of polynomials of degrees less than or equal to 1. Let ϱ_k be the following restriction of this map.

Definition 1. For any polynomial $P(x)$, let f be the map that maps $P(x)$ to $R(x)$ in (4.6). Then the map ϱ_k is the restriction of f to the linear subspace spanned by the orthogonal polynomials L_k, L_{k-1} .

We then have the following.

Lemma 1. If $\langle L_k, L_{k-1} \rangle_1 \neq 0$, then the map ϱ_k is invertible.

Proof. Suppose there is exists non-zero constants a_1 and a_2 such that

$$a_1 L_k + a_2 L_{k-1} = \frac{d}{dx} (q(x)x(t - \tilde{\tau}x)w) w^{-1}$$

for some polynomial $q(x)$ of degree $k - 2$, then by taking the skew product $\langle \rangle_1$ of this polynomial with L_k , we obtain

$$a_2 \langle L_{k-1}, L_k \rangle_1 = \langle a_1 L_k + a_2 L_{k-1}, L_k \rangle_1 = \langle q(x), L_k \rangle_2 = 0.$$

As $q(x)$ is of degree $k - 2$. Since $\langle L_{k-1}, L_k \rangle_1 \neq 0$, this shows that $a_2 = 0$. By taking the skew product with L_{k-1} , we conclude that $a_1 = 0$ and hence the map ϱ_k has a trivial kernel. \square

In particular, we have the following.

Corollary 1. If k is even, then $\langle L_k, L_{k-1} \rangle_1 = 0$.

Proof. Let $q(x)$ be a polynomial of degree $k - 2$ that satisfies the following conditions

$$\int_{\mathbb{R}_+} \frac{d}{dx} (q(x)w_4(x))x^j w_4(x) dx = 0, \quad j = 0, \dots, k - 2, \quad (4.7)$$

where $w_4(x) = x^{\frac{M-N+1}{2}}(t - \tilde{\tau}x)^{\frac{1}{2}}e^{-\frac{Mx}{2}}$. A non trivial polynomial $q(x)$ of degree $k - 2$ that satisfies these conditions exists if and only if the moment matrix with entries $\int_{\mathbb{R}_+} \frac{d}{dx} (x^i w_4(x))x^j w_4(x) dx$ has a vanishing determinant. For even k , the moment matrix is of odd dimension and anti-symmetric and hence its determinant is always zero.

Assuming k is even and let $q(x)$ be a polynomial that satisfies (4.7). By taking the inner product $\langle \rangle_2$ with x^j , we see that there exists non-zero constants a_1 and a_2 such that

$$a_1 L_k + a_2 L_{k-1} = \frac{d}{dx} (q(x)x(t - \tilde{\tau}x)w)w^{-1},$$

Therefore by Lemma 1, we see that if k is even, we will have $\langle L_k, L_{k-1} \rangle_1 = 0$. \square

Lemma 1 shows that if $\langle L_i, L_{i-1} \rangle_1 \neq 0$, then there exists two independent polynomials $R_0(y)$ and $R_1(y)$ in the span of y and 1 such that $R_j(y) = \varrho_i(L_{i-j})$. Then we have

$$R_j(y) = -\frac{d}{dy} (q_j(y)y(t - \tilde{\tau}y)w)w^{-1} + L_{i-j}(y), \quad j = 0, 1.$$

In particular, the skew product of $R_j(y)$ with L_{i-l} , $l < 2$ is given by

$$\langle L_{i-l}(x), R_j(y) \rangle_1 = -\langle L_{i-l}q_j \rangle_2 + \langle L_{i-l}, L_{i-j} \rangle_1.$$

As q_j is a polynomial of degree less than or equal to $i - 2$ and $l < 2$, the first term on the right hand side is zero. Therefore we have

$$\langle L_{i-l}(x), R_j(y) \rangle_1 = \langle L_{i-l}, L_{i-j} \rangle_1, \quad l < 2, \quad j = 0, 1. \quad (4.8)$$

We can now show that the skew orthogonal polynomials $\pi_{2k,1}$ and $\pi_{2k+1,1}$ exist if $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$.

Proposition 4. *If $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$, then the skew orthogonal polynomials $\pi_{2k,1}$ and $\pi_{2k+1,1}$ both exist and $\pi_{2k,1}$ is unique while $\pi_{2k+1,1}$ is unique up to an addition of a multiple of $\pi_{2k,1}$. Moreover, we have $\langle L_{2k}, L_{2k-1} \rangle_1 = 0$ and the skew orthogonal polynomials are given by*

$$\begin{aligned} \pi_{2k,1} &= L_{2k} - \frac{\langle L_{2k}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1}, \\ \pi_{2k+1,1} &= L_{2k+1} - \frac{\langle L_{2k+1}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-1} + \frac{\langle L_{2k+1}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1} L_{2k-2} + c\pi_{2k,1}, \end{aligned} \quad (4.9)$$

for $k \geq 2$, where c is an arbitrary constant.

Proof. Let $\pi_{2k,1}$ and $\pi_{2k+1,1}$ be polynomials defined by

$$\begin{aligned}\pi_{2k,1}(x) &= L_{2k}(x) + \gamma_{2k,1}L_{2k-1}(x) + \gamma_{2k,2}L_{2k-2}(x), \\ \pi_{2k+1,1}(x) &= L_{2k+1}(x) + \gamma_{2k+1,1}L_{2k-1}(x) + \gamma_{2k+1,2}L_{2k-2}(x),\end{aligned}$$

for some constants $\gamma_{j,k}$. If we can show that $\langle \pi_{2k-l,1}, y^j \rangle_1 = 0$ for $j = 0, 1$ and $l = -1, 0$, then $\pi_{2k-l,1}$ will be the skew orthogonal polynomial. Let R_0 and R_1 be the images of L_{2k-1} and L_{2k-2} under the map ϱ_{2k-1} . Then by the assumption in the Proposition, they are independent in the span of y and 1. Therefore the conditions $\langle \pi_{2k-l,1}, y^j \rangle_1 = 0$ are equivalent to $\langle \pi_{2k-l,1}, R_j(y) \rangle_1 = 0$. By taking $i = 2k - 1$ in (4.8), we see that this is equivalent to $\langle \pi_{2k-l,1}, L_{2k-1-j} \rangle_1 = 0$. This implies

$$\begin{aligned}\langle \pi_{2k,1}, L_{2k-1} \rangle_1 &= \langle L_{2k}, L_{2k-1} \rangle_1 + \gamma_{2k,2} \langle L_{2k-2}, L_{2k-1} \rangle_1 = 0, \\ \langle \pi_{2k,1}, L_{2k-2} \rangle_1 &= \langle L_{2k}, L_{2k-2} \rangle_1 + \gamma_{2k,1} \langle L_{2k-1}, L_{2k-2} \rangle_1 = 0.\end{aligned}$$

Hence we have

$$\gamma_{2k,1} = -\frac{\langle L_{2k}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1}, \quad \gamma_{2k,2} = \frac{\langle L_{2k}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1},$$

which exist and are unique as $\langle L_{2k-1}, L_{2k-2} \rangle_1 \neq 0$. This determines $\pi_{2k,1}$ uniquely. By Corollary 1, we have $\langle L_{2k}, L_{2k-1} \rangle_1 = 0$ and hence $\gamma_{2k,2} = 0$. Similarly, the coefficients for $\pi_{2k+1,1}$ are

$$\gamma_{2k+1,1} = -\frac{\langle L_{2k+1}, L_{2k-2} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1}, \quad \gamma_{2k+1,2} = \frac{\langle L_{2k+1}, L_{2k-1} \rangle_1}{\langle L_{2k-1}, L_{2k-2} \rangle_1}.$$

Again, these coefficients exist and are unique. However, as $\langle \pi_{2k,1}, \pi_{2k+1,1} \rangle_1 = 0$ and $\langle \pi_{2k,1}, y^j \rangle_1 = 0$ for $j = 0, \dots, 2k - 1$, adding any multiple of $\pi_{2k,1}$ to $\pi_{2k+1,1}$ will not change the orthogonality conditions $\langle \pi_{2k+1,1}, y^j \rangle_1 = 0$ that is satisfied by $\pi_{2k+1,1}$ and hence $\pi_{2k+1,1}$ is only determined up to the addition of a multiple of $\pi_{2k,1}$. \square

5 The Christoffel Darboux formula for the kernel

In [26], skew orthogonal polynomials were interpreted as multi-orthogonal polynomials and represented as the solution of a Riemann-Hilbert problem. This representation allows us to use the results in [15] to derive a Christoffel-Darboux formula for the kernel (1.9) in terms of the Riemann-Hilbert problem.

Let us recall the definitions of multi-orthogonal polynomials. First let the weights w_0 , w_1 and w_2 be

$$w_0(x) = x^{M-N} e^{-Mx}, \quad w_l(x) = w(x) \int_{\mathbb{R}_+} \epsilon(x-y) L_{N-l-2}(y) w(y) dy, \quad l = 1, 2.$$

Note that the weights $w_l(x)$ are defined with the polynomials L_{N-3} and L_{N-4} instead of L_{N-1} and L_{N-2} . This is because the construction below involves the polynomial $\pi_{N-2,1}$ as

well as $\pi_{N,1}$. By taking $i = N - 3$ in (4.8), we see that the orthogonality conditions for $\pi_{N,1}$ is also equivalent to

$$\begin{aligned}\langle \pi_{N,1}, x^j \rangle_2 &= 0, \quad j = 0, \dots, N - 3, \\ \langle \pi_{N,1}, L_{N-j} \rangle_2 &= 0, \quad j = 3, 4,\end{aligned}$$

provided $\langle L_{N-3}, L_{N-4} \rangle_1$ is also non-zero.

Then the orthogonol polynomials of type II $P_{N,l}^{II}(x)$, [2], [3], [9], [16] are polynomials of degree $N - 1$ such that

$$\begin{aligned}\int_{\mathbb{R}_+} P_{N,l}^{II}(x) x^j w_0(x) dx &= 0, \quad 0 \leq j \leq N - 3, \\ \int_{\mathbb{R}_+} P_{N,l}^{II}(x) w_m(x) dx &= -2\pi i \delta_{lm}, \quad l, m = 1, 2.\end{aligned}\tag{5.1}$$

Remark 2. *More accurately, these are in fact the multi-orthogonal polynomials with indices $(N - 2, \vec{e}_l)$, where $\vec{e}_1 = (0, 1)$ and $\vec{e}_2 = (1, 0)$.*

We will now define the multi-orthogonal polynomials of type I. Let $P_{N,l}^I(x)$ be a function of the following form

$$P_{N,l}^I(x) = B_{N,l}(x)w_0 + \sum_{k=1}^2 (\delta_{kl}x + A_{N,k,l}) w_k,\tag{5.2}$$

where $B_{N,l}(x)$ is a polynomial of degree $N - 3$ and $A_{N,k,l}$ independent on x . Moreover, let $P_{N,l}^I(x)$ satisfies

$$\int_{\mathbb{R}_+} P_{N,l}^I(x) x^j dx = 0, \quad j = 0, \dots, N - 1.$$

Then the polynomials $B_{N,l}(x)$ and $\delta_{kl}x + A_{N,k,l}$ are multi-orthogonal polynomials of type I with indices $(N, 2, 1)$ for $l = 1$ and $(N, 1, 2)$ for $l = 2$. We will now show that $P_{N,1}^{II}$ and $P_{N,2}^{II}$ exist and are unique if both $\langle L_{N-1}, L_{N-2} \rangle_1$ and $\langle L_{N-3}, L_{N-4} \rangle_1$ are non-zero.

Lemma 2. *Suppose both $\langle L_{N-1}, L_{N-2} \rangle_1$ and $\langle L_{N-3}, L_{N-4} \rangle_1$ are non-zero, then the polynomials $P_{N,1}^{II}$ and $P_{N,2}^{II}$ exist and are unique.*

Proof. Let us write L_{N-l} as

$$L_{N-l} = \frac{d}{dx} (q_l(x)x(t - \tilde{\tau}x)w) w^{-1} + R_l(x), \quad l = 1, \dots, 4.$$

for some polynomials q_l of degree $N - l - 2$ and R_l of degree 1, then by Lemma 1, we see that both ϱ_{N-1} and ϱ_{N-3} are invertible and hence the composition $\varrho_{N-1}\varrho_{N-3}^{-1}$ is also invertible. In particular, there exists a 2×2 invertible matrix with entries $c_{l,k}$ and polynomials \tilde{q}_l of degree $N - 3$ such that

$$L_{N-l} = \frac{d}{dx} (\tilde{q}_l(x)x(t - \tilde{\tau}x)w) w^{-1} + c_{l-2,1}L_{N-1} + c_{l-2,2}L_{N-2}, \quad l = 3, 4.$$

Then by the first condition in (5.1), we see that

$$\int_{\mathbb{R}_+} P_{N,l}^{II} w_j(x) dx = \langle P_{N,l}^{II}, c_{l,1} L_{N-1} + c_{l,2} L_{N-2} \rangle_1 = -2\pi i \delta_{lj}, \quad l = 1, 2, \quad j = 1, 2. \quad (5.3)$$

As the matrix with entries c_{ij} is invertible, we see that the linear equations (5.3) has a unique solution in the linear span of L_{N-1} and L_{N-2} if and only if $\langle L_{N-1}, L_{N-2} \rangle_1 \neq 0$. \square

As we shall see, existence and uniqueness of $P_{N,l}^{II}$ would imply that the multi-orthogonal polynomials of type I also exist and are unique. As in [26], the multi-orthogonal polynomial together with the skew orthogonal polynomials form the solution of a Riemann-Hilbert problem. Let $Y(x)$ be the matrix

$$Y(x) = \begin{pmatrix} \pi_{N,1}(x) & C(\pi_{N,1} w_0) & C(\pi_{N,1} w_1) & C(\pi_{N,1} w_2) \\ \kappa \pi_{N-2,1}(x) & \cdots & \cdots & \cdots \\ P_{N,1}^{II}(x) & \ddots & \cdots & \cdots \\ P_{N,2}^{II}(x) & \cdots & \cdots & \cdots \end{pmatrix}, \quad (5.4)$$

where κ is the constant

$$\kappa = -\frac{2\pi i}{\langle \pi_{N-2,1}, x^{N-3} \rangle_2} = -\frac{4\pi i}{\tilde{\tau} M h_{N-1,1}}, \quad h_{2j-1,1} = \langle \pi_{2j-2,1}, \pi_{2j-1,1} \rangle_1. \quad (5.5)$$

and $C(f)$ is the Cauchy transform

$$C(f)(x) = \frac{1}{2\pi i} \int_{\mathbb{R}_+} \frac{f(s)}{s-x} ds. \quad (5.6)$$

Then by using the orthogonality conditions of the skew orthogonal polynomials and multi-orthogonal polynomials, together with the jump discontinuity of the Cauchy transform, one can check that $Y(x)$ satisfies the following Riemann-Hilbert problem.

1. $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$,
2. $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_0 & w_1(z) & w_2(z) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}_+,$
3. $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^N & & & \\ & z^{-N+2} & & \\ & & z^{-1} & \\ & & & z^{-1} \end{pmatrix}, \quad z \rightarrow \infty.$

The multi-orthogonal polynomials of type I can also be arranged to satisfy a Riemann-Hilbert problem. Let ϵ be the operator

$$\epsilon(f)(x) = \frac{1}{2} \int_0^\infty \epsilon(x-y) f(y) dy. \quad (5.8)$$

First note that the functions $\psi_j(x) = \epsilon(\pi_{j,1}w)$ for $j \geq 2$, can be express in the form of (5.2). By Lemma 1, we can write $\pi_{j,1}(x)$ as

$$\pi_{j,1}(x) = \frac{d}{dx} (B_j(x)x(t - \tilde{\tau}x)w) w^{-1} + A_{j,1}L_{N-3} + A_{j,2}L_{N-4},$$

Then we have

$$\epsilon(\pi_{j,1}w)w = B_j(x)w_0 + A_{j,1}w_1 + A_{j,2}w_2. \quad (5.9)$$

where $B_j(x)$ is a polynomial of degree $j - 2$.

Let $X(z)$ be the matrix value function defined by

$$X(z) = \begin{pmatrix} -\frac{\kappa\tilde{\tau}M}{2}C(\psi_{N-2}w) & \frac{\kappa\tilde{\tau}M}{2}B_{N-2} & \frac{\kappa\tilde{\tau}M}{2}A_{N-2,1} & \frac{\kappa\tilde{\tau}M}{2}A_{N-2,2} \\ -\frac{\tilde{\tau}M}{2}C(\psi_Nw) & \frac{\tilde{\tau}M}{2}B_N & \frac{\tilde{\tau}M}{2}A_{N,1} & \frac{\tilde{\tau}M}{2}A_{N,2} \\ -C(P_{N,1}^I) & \dots & \dots & \dots \\ -C(P_{N,2}^I) & \dots & \dots & \dots \end{pmatrix}. \quad (5.10)$$

Then by using the orthogonality and the the jump discontinuity of the Cauchy transform, it is easy to check that $X^{-T}(z)$ and $Y(z)$ satisfies the same Riemann-Hilbert problem and hence the multi-orthogonal polynomials of type I also exist and are unique.

We will now show that the kernel $S_1(x, y)$ given by (1.9) can be expressed in terms of the matrix $Y(z)$.

Proposition 5. *Suppose $\langle L_{N-3}, L_{N-4} \rangle_1 \langle L_{N-1}, L_{N-2} \rangle_1 \neq 0$ and let the kernel $S_1(x, y)$ be*

$$S_1(x, y) = - \sum_{j,k=0}^{N-1} r_j(x)w(x)\mu_{jk}\epsilon(r_kw)(y), \quad (5.11)$$

where $r_j(x)$ is an arbitrary degree j monic polynomial for $j < N - 2$ and $r_j(x) = \pi_{j,1}(x)$ for $j \geq N - 2$. The matrix μ with entries μ_{jk} is the inverse of the matrix \mathcal{M} whose entries are given by

$$(\mathcal{M})_{jk} = \langle r_j, r_k \rangle_1, \quad j, k = 0, \dots, N - 1. \quad (5.12)$$

Then the kernel $S_1(x, y)$ exists and is equal to

$$S_1(x, y) = \frac{w(x)w^{-1}(y)}{2\pi i(x - y)} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & w_0(y) & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y)Y_+(x). \quad (5.13)$$

Proof. First note that, since $\langle L_{N-1}, L_{N-2} \rangle_1 \neq 0$ and $\langle L_{N-3}, L_{N-4} \rangle_1 \neq 0$, the skew orthogonal polynomials $\pi_{N-l,1}$ exist for $l = -1, \dots, 2$. In particular, the moment matrix $\tilde{\mathcal{M}}$ with entries

$$(\tilde{\mathcal{M}})_{jk} = \langle x^j, y^k \rangle_1, \quad j, k = 0, \dots, N - 1$$

is invertible. Since the polynomials $\pi_{N-j,1}$ exist for $j = -1, \dots, 2$, the sequence $r_k(x)$ and x^k are related by an invertible transformation. Therefore the matrix \mathcal{M} in (5.12) is also invertible. As the matrix \mathcal{M} is of the form

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_2 & 0 \\ 0 & h_{N-1,1}\mathcal{J} \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where \mathcal{M}_2 has entries $\langle r_j, r_k \rangle_1$ for j, k from 0 to $N-3$. From this, we see that the matrix μ_{jk} is of the form

$$\mu = \begin{pmatrix} \mathcal{M}_2^{-1} & 0 \\ 0 & -h_{N-1,1}^{-1}\mathcal{J} \end{pmatrix}. \quad (5.14)$$

As in [15], let us now expand the functions $xr_j(x)$ and $x\epsilon(r_j w)$.

$$\begin{aligned} xr_j(x) &= \sum_{k=0}^{N-1} c_{jk} r_k(x) + \delta_{N-1,j} \pi_{N,1}(x), \\ x\epsilon(r_j w)(x) &= \sum_{k=0}^{N-1} d_{jk} \epsilon(r_k w) + d_{j,N} \psi_N + d_{j,N+1} \frac{P_{N,1}^I(x)}{w(x)} + d_{j,N+2} \frac{P_{N,2}^I(x)}{w(x)}. \end{aligned} \quad (5.15)$$

Then the coefficients c_{jk} and d_{jk} for $j, k = 0, \dots, N-1$ are given by

$$c_{jk} = \sum_{l=0}^{N-1} \langle xr_j, r_l \rangle_1 \mu_{lk}, \quad d_{jk} = \sum_{l=0}^{N-1} \mu_{kl} \langle xr_l, r_k \rangle_1.$$

Therefore if we let \mathcal{C} be the matrix with entries c_{jk} , $j, k = 0, \dots, N-1$ and \mathcal{D} be the matrix with entries d_{jk} for $j, k = 0, \dots, N-1$, then we have

$$\mathcal{C} = \mathcal{M}_1 \mu, \quad \mathcal{D} = (\mu \mathcal{M}_1)^T, \quad (5.16)$$

where \mathcal{M}_1 is the matrix with entries $\langle xr_j, r_k \rangle_1$ for $j, k = 0, \dots, N-1$. From (5.15), we obtain

$$\begin{aligned} (y-x)S_1(x, y) &= \sum_{j=0}^{N-1} \pi_{N,1}(x) w(x) \mu_{N-1,j} \epsilon(r_j w)(y) \\ &\quad - \sum_{j,k=0}^{N-1} r_j(x) w(x) \mu_{jk} \left(d_{k,N} \psi_N(y) + d_{k,N+1} \frac{P_{N,1}^I(y)}{w(y)} + d_{k,N+2} \frac{P_{N,2}^I(y)}{w(y)} \right) \\ &\quad + r^T(x) w(x) (\mathcal{C}^T \mu - \mu \mathcal{D}) \epsilon(rw)(y), \end{aligned} \quad (5.17)$$

where $r(x)$ is the column vector with components $r_k(x)$. Now by (5.16), we see that

$$\mathcal{C}^T \mu = \mu^T \mathcal{M}_1^T \mu, \quad \mu \mathcal{D} = \mu \mathcal{M}_1^T \mu^T,$$

which are equal as $\mu^T = -\mu$. Now from the form of μ in (5.14), we see that $\mu_{N-1,j} = \delta_{j,N-2} h_{N-1,1}^{-1}$. Hence we have

$$\sum_{j=0}^{N-1} \pi_{N,1}(x) w(x) \mu_{N-1,j} \epsilon(r_j w)(y) = h_{N-1,1}^{-1} \pi_{N,1}(x) w(x) \psi_{N-2}(y). \quad (5.18)$$

Let us now consider the second term in (5.17). As in (5.9) we can write $\epsilon(r_k w)w$ as

$$\epsilon(r_k w)w = q_k(x) + D_{k,1}w_1 + D_{k,2}w_2,$$

Then from the form of $P_{N,j}^I$ in (5.2) and the orthogonality condition (5.1), we see that the coefficients $d_{k,N+l}$, $l = 1, 2$ are given by

$$d_{k,N+l} = D_{k,l} = -\frac{1}{2\pi i} \int_{\mathbb{R}_+} P_{N,l}^{II}(x) \epsilon(r_k w) w dx.$$

For $k \neq N-1$, the polynomial q_k is of degree less than or equal to $N-4$, while B_N is a polynomial of degree $N-2$, therefore the coefficient $d_{k,N}$ is zero unless $k = N-1$. For $k = N-1$, it is given by the leading coefficient of q_{N-1} divided by the leading coefficient of B_N . Since

$$\pi_{N-1,1}(x) = \frac{d}{dx} (q_{N-1}(x) x(t - \tilde{\tau}x) w) w^{-1} + D_{k,1}L_{N-3} + D_{k,2}L_{N-4},$$

we see that both the leading coefficient of $q_{N-1}(x)$ and B_N is $\frac{2}{M\tilde{\tau}}$. Hence $d_{k,N}$ is $\delta_{k,N-1}$. This gives us

$$\sum_{k,j=0}^{N-1} r_j(x) w(x) \mu_{jk} d_{k,N} \psi_N(y) = -h_{N-1,1}^{-1} \pi_{N-2,1}(x) w(x) \psi_N(y).$$

To express the second term in (5.17) in terms of the multi-orthogonal polynomials, let us now express $P_{N,l}^{II}$ in terms of the polynomials r_k . Let us write $P_{N,l}^{II} = \sum_{j=0}^{N-1} a_j r_j(x)$. Then we have

$$\int_{\mathbb{R}_+} P_{N,l}^{II}(x) \epsilon(r_k w) w dx = \sum_{j=0}^{N-1} a_j (\mathcal{M}_1)_{jk}, \quad \sum_{k=0}^{N-1} \int_{\mathbb{R}_+} P_{N,l}^{II}(x) \epsilon(r_k w) w dx \mu_{kj} = a_j.$$

Hence $P_{N,l}^{II}(x)$ can be written as

$$P_{N,l}^{II}(x) = \sum_{k,j=0}^{N-1} \left(\int_{\mathbb{R}_+} P_{N,l}^{II}(x) \epsilon(r_k w) w dx \mu_{kj} \right) r_j(x) = -2\pi i \sum_{k,j=0}^{N-1} d_{k,N+l} \mu_{kj} r_j(x)$$

Therefore the second term in (5.17) is given by

$$\begin{aligned} & \sum_{j,k=0}^{N-1} r_j(x)w(x)\mu_{jk} \left(d_{k,N}\psi_N(y) + d_{k,N+1}\frac{P_{N,1}^I(y)}{w(y)} + d_{k,N+2}\frac{P_{N,2}^I(y)}{w(y)} \right) \\ &= -h_{N-1,1}^{-1}\pi_{N-2,1}(x)\psi_N(y) + \frac{1}{2\pi i} \sum_{l=1}^2 P_{N,l}^{II}(x)w(x)w^{-1}(y)P_{N,l}^I(y). \end{aligned}$$

From this and (5.18), we obtain

$$\begin{aligned} (y-x)S_1(x,y) &= h_{N-1,1}^{-1}\pi_{N,1}(x)w(x)\psi_{N-2}(y) + h_{N-1,1}^{-1}\pi_{N-2,1}(x)w(x)\psi_N(y) \\ &\quad - \frac{1}{2\pi i} \sum_{l=1}^2 P_{N,l}^{II}(x)w(x)w^{-1}(y)P_{N,l}^I(y). \end{aligned}$$

By using the fact that $Y^{-1}(y) = X^T(y)$ and the expressions of the matrix Y (5.4) and X (5.10), together with (5.5), we see that this is the same as (5.13). \square

5.1 The kernel in terms of Laguerre polynomials

We will now use a result in [6] to further simplify the expression of the kernel $S_1(x,y)$ so that its asymptotics can be computed using the asymptotics of Laguerre polynomials. Let us recall the set up in [6]. First let $Y(x)$ be a matrix satisfying the Riemann-Hilbert problem

1. $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$,
2. $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_0(z) & w_1(z) & \cdots & w_r(z) \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & & \cdots & & 1 \end{pmatrix}, \quad z \in \mathbb{R}_+$
3. $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & & & & \\ & z^{-n+r} & & & \\ & & z^{-1} & & \\ & & & \ddots & \\ & & & & z^{-1} \end{pmatrix}, \quad z \rightarrow \infty.$

and let $\mathcal{K}_1(x,y)$ be the kernel given by

$$\mathcal{K}_1(x,y) = \frac{w_0(x)w_0^{-1}(y)}{2\pi i(x-y)} \begin{pmatrix} 0 & w_0(y) & \cdots & w_r(y) \end{pmatrix} Y_+^{-1}(y)Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (5.19)$$

Let $\pi_{j,2}(x)$ be the monic orthogonal polynomials with respect to the weight $w_0(x)$. Let $\mathcal{K}_0(x, y)$ be the following kernel.

$$\mathcal{K}_0(x, y) = w_0(x) \frac{\pi_{2,n}(x)\pi_{2,n-1}(y) - \pi_{2,n}(y)\pi_{2,n-1}(x)}{h_{n-1,2}(x-y)}, \quad (5.20)$$

where $h_{j,2} = \int_0^\infty \pi_{j,2}^2 w_0 dx$. Let $\pi(z)$, $v(z)$ and $u(z)$ be the following vectors

$$\pi(z) = (\pi_{n-r,2}, \dots, \pi_{n-1,2})^T, \quad v(z) = (w_1, \dots, w_r)^T w_0^{-1}, \quad u(z) = (I - \mathcal{K}_0^T) v(z) \quad (5.21)$$

and let B be the matrix $B = \int_{\mathbb{R}_+} \pi(z) v^T(z) w_0(z) dz$. Then the kernel $\mathcal{K}_1(x, y)$ can be express as [6]

Proposition 6. *Suppose $\int_{\mathbb{R}_+} p(x) w_i(x) dx$ converges for any polynomial $p(x)$. Then the kernel $\mathcal{K}_1(x, y)$ defined by (5.19) is given by*

$$\mathcal{K}_1(x, y) - \mathcal{K}_0(x, y) = w_0(x) u^T(y) B^{-1} \pi(x). \quad (5.22)$$

Remark 3. *Although in [6], the theorem is stated with the jump of $Y(x)$ on \mathbb{R} instead of \mathbb{R}_+ , while the weights are of the special form $w_0 = e^{-NV(x)}$, $w_j(x) = e^{a_j x}$, where $V(x)$ is an even degree polynomial, the proof in [6] in fact remain valid as long as integrals of the form $\int_{\mathbb{R}_+} p(x) w_i(x) dx$ converges for any polynomial $p(x)$. This is true in our case.*

We can now apply Proposition 6 to our case. In our case, the vectors $\pi(z)$ and $v(z)$ are given by

$$\pi(z) = (L_{N-2} \quad L_{N-1})^T, \quad v(z) = (w_1 \quad w_2)^T w_0^{-1},$$

while the matrix B is given by

$$B = \begin{pmatrix} \langle L_{N-2}, L_{N-3} \rangle_1 & \langle L_{N-2}, L_{N-4} \rangle_1 \\ \langle L_{N-1}, L_{N-3} \rangle_1 & \langle L_{N-1}, L_{N-4} \rangle_1 \end{pmatrix}$$

By corollary 1, we see that $\langle L_{N-2}, L_{N-3} \rangle_1 = 0$ and hence the determinant of B is

$$\det B = \langle L_{N-2}, L_{N-4} \rangle_1 \langle L_{N-1}, L_{N-3} \rangle_1.$$

From Lemma 2, we see that B is invertible if and only if the multi-orthogonal polynomials $P_{N,l}^{II}$ exist. Let us now consider the vector $u(z)$. It is given by

$$u(z) = (I - \mathcal{K}_0^T) v(z) = v(z) - \sum_{j=0}^{N-1} \frac{L_j(z)}{h_{j,0}} (\langle L_j, L_{N-3} \rangle_1 \quad \langle L_j, L_{N-4} \rangle_1)$$

by the Christoffel-Darboux formula, where $h_{j,0} = \langle L_j, L_j \rangle_2$. We shall show that L_{N-3} and L_{N-4} can be written in the following form

$$L_{N-l} = \frac{d}{dx} (q_l(x) x(t - \tilde{\tau}x) w) w^{-1} + C_{l-2,1} \pi_{N+1,1} + C_{l-2,2} \pi_{N,1}, \quad l = 3, 4, \quad (5.23)$$

for some polynomial $q_l(x)$ of degree $N - 1$. By Lemma 1, we see that if $\langle L_{N-3}, L_{N-4} \rangle_1 \neq 0$, then the map ϱ_{N-3} in Definition 1 is invertible. Therefore if the restriction of the map f in Definition 1 is also invertible on the span of $\pi_{N+1,1}$ and $\pi_{N,1}$, we will be able to write L_{N-3} and L_{N-4} in the form of (5.23).

Lemma 3. *Let f be the map in Definition 1 and let ϱ_π be its restriction to the span of $\pi_{N+1,1}$ and $\pi_{N,1}$. Then ϱ_π is invertible.*

Proof. Suppose there exist a_1 and a_2 such that

$$a_1\pi_{N+1,1} + a_2\pi_{N,1} = \frac{d}{dx} (q(x)x(t - \tilde{\tau}x)w) w^{-1}$$

for some polynomial $q(x)$ of degree at most $N - 1$. Then we have

$$\langle x^j, a_1\pi_{N+1,1} + a_2\pi_{N,1} \rangle_1 = \langle x^j q(x) \rangle_2 = 0, \quad j = 0, \dots, N - 1.$$

As the degree of $q(x)$ is at most $N - 1$, this is only possible if $q(x) = 0$. \square

The composition of ϱ_{N-3} and ϱ_π^{-1} will therefore give us a representation of L_{N-3} and L_{N-4} in the form of (5.23). By using this representation and the fact that $q_l(x)$ is of degree at most $N - 1$, we see that

$$\begin{aligned} \mathcal{K}_0^T (w_l w_0^{-1}) &= \sum_{j=0}^{N-1} \frac{L_j(x)}{h_{j,0}} \langle L_j, L_{N-l-2} \rangle_1 \\ &= \sum_{j=0}^{N-1} \frac{L_j(x)}{h_{j,0}} (\langle L_j, q_l \rangle_2 + \langle L_j, C_{l-2,1}\pi_{N+1,1} + C_{l-2,2}\pi_{N,1} \rangle_1), \\ &= q_l(x). \end{aligned}$$

Therefore the vector $u(x)$ is given by

$$u(x) = w(x)w_0^{-1}(x)C\epsilon(\pi_{N+1,1}w \quad \pi_{N,1}w)^T$$

where C is the matrix with entries $C_{i,j}$. We will now determine the constants $C_{i,j}$.

Lemma 4. *Let $\pi_{N,1}$ and $\pi_{N+1,1}$ be the monic skew orthogonal polynomial with respect to the weight $w(x)$ and choose $\pi_{N+1,1}$ so that the constant c in (4.9) is zero. Then the vector $u(y)$ in (5.22) is given by*

$$u(x) = w(x)w_0^{-1}(x)C\epsilon(\pi_{N+1,1}w \quad \pi_{N,1}w)^T, \quad (5.24)$$

where C is the matrix whose entries $C_{i,j}$ are given by

$$\begin{aligned} C_{i,1} &= -\frac{M\tilde{\tau} \langle L_{N-1}, L_{N-i-2} \rangle_1}{2h_{N-1,0}}, \\ C_{i,2} &= (Mt - \tilde{\tau}(N + M)) \frac{\langle L_{N-1}, L_{N-i-2} \rangle_1}{2h_{N-1,0}} - M\tilde{\tau} \frac{\langle L_{N-2}, L_{N-i-2} \rangle_1}{2h_{N-2,0}}. \end{aligned} \quad (5.25)$$

Proof. First let us compute the leading order coefficients of the polynomial $q_l(x)$ in (5.23). Let $q_l(x) = q_{l,N-1}x^{N-1} + q_{l,N-2}x^{N-2} + O(x^{N-3})$, then we have

$$\begin{aligned} \frac{d}{dx} (q_l(x)x(t - \tilde{\tau}x)w) w^{-1} &= \frac{M\tilde{\tau}}{2}q_{l,N-1}x^{N+1}, \\ &+ \left(-\frac{\tilde{\tau}}{2}(N+M)q_{l,N-1} - \frac{Mt}{2}q_{l,N-1} + \frac{M\tilde{\tau}}{2}q_{l,N-2} \right) x^N + O(x^{N-1}) \end{aligned}$$

From (5.23), we see that

$$q_{l,N-1} = -\frac{2}{M\tilde{\tau}}C_{l-2,1}. \quad (5.26)$$

On the other hand, by orthogonality, we have

$$\langle L_{N-1}, L_{N-l} \rangle_1 = \langle L_{N-1}, q_l \rangle_2 = q_{l,N-1}h_{N-1,0}.$$

Therefore $C_{l-2,1}$ is given by

$$C_{l-2,1} = -\frac{M\tilde{\tau} \langle L_{N-1}, L_{N-l} \rangle_1}{2h_{N-1,0}}$$

Let us now compute $C_{i,2}$. By taking the skew product, we have

$$\langle L_{N-2}, L_{N-l} \rangle_1 = q_{l,N-1} \langle L_{N-2}, x^{N-1} \rangle_2 + q_{l,N-2}h_{N-2,0}$$

Now from (4.5), we obtain

$$\begin{aligned} \langle x^{j+1}, L_j \rangle_2 &= \left\langle L_{j+1} + \frac{(M-N+j+1)(j+1)}{M}x^j, L_j \right\rangle_2, \\ &= \frac{(M-N+j+1)(j+1)}{M}h_{j,0}. \end{aligned} \quad (5.27)$$

Hence $q_{l,N-2}$ is equal to

$$q_{l,N-2} = \frac{\langle L_{N-2}, L_{N-l} \rangle_1}{h_{N-2,0}} - q_{l,N-1} \frac{(M-1)(N-1)}{M} \quad (5.28)$$

By using the expansion (4.9) of the skew orthogonal polynomials in terms of L_{N-k} , we have

$$\begin{aligned} \langle L_{N-l}, L_N \rangle_2 &= \frac{M\tilde{\tau}}{2}q_{l,N-1} \langle x^{N+1}, L_N \rangle_2 + C_{l-2,2}h_{N,0} \\ &+ \left(\frac{\tilde{\tau}}{2}(-N-M)q_{l,N-1} - \frac{Mt}{2}q_{l,N-1} + \frac{M\tilde{\tau}}{2}q_{l,N-2} \right) h_{N,0} \end{aligned}$$

By substituting (5.28) into this, we obtain

$$C_{l-2,2} = \left(-\frac{\tilde{\tau}}{2}(M+N) + \frac{Mt}{2} \right) \frac{\langle L_{N-1}, L_{N-l} \rangle_1}{h_{N-1,0}} - \frac{M\tilde{\tau}}{2} \frac{\langle L_{N-2}, L_{N-l} \rangle_1}{h_{N-2,0}}$$

This proves the lemma. \square

From this and (5.22), we obtain the following.

Corollary 2. *The kernel $S_1(x, y)$ defined by (5.11) is given by*

$$S_1(x, y) - K_2(x, y) = \epsilon \begin{pmatrix} \pi_{N+1,1} w & \pi_{N,1} w \end{pmatrix} (y) \begin{pmatrix} 0 & -\frac{M\tilde{\tau}}{2h_{N-1}} \\ -\frac{M\tilde{\tau}}{2h_{N-2}} & \frac{Mt-\tilde{\tau}(N+M)}{2h_{N-1}} \end{pmatrix} \pi(x) w(x) \quad (5.29)$$

where $K_2(x, y)$ is the kernel of the Laguerre polynomials

$$K_2(x, y) = \left(\frac{y(t - \tilde{\tau}y)}{x(t - \tilde{\tau}x)} \right)^{\frac{1}{2}} w_0^{\frac{1}{2}}(x) w_0^{\frac{1}{2}}(y) \frac{L_N(x)L_{N-1}(y) - L_N(y)L_{N-1}(x)}{h_{N-1,0}(x - y)} \quad (5.30)$$

6 Derivative of the partition function

In this section we will derive a formula for the derivative of determinant of the matrix \mathcal{M} given in (5.12). We have the following.

Proposition 7. *Let \mathcal{M} be the matrix given by (5.12), where the sequence of monic polynomials $r_j(x)$ in (5.12) is chosen such that $r_j(x)$ are arbitrary degree j monic polynomials that are independent on t and $r_j(x) = \pi_{j,1}(x)$ for $j = N - 2, N - 1$. Then the logarithmic derivative of $\det \mathcal{M}$ with respect to t is given by*

$$\frac{\partial}{\partial t} \log \det \mathcal{M} = \int_{\mathbb{R}_+} \frac{S_1(x, x)}{t - \tilde{\tau}x} dx, \quad (6.1)$$

where $S_1(x, y)$ is the kernel given in (5.11).

Proof. First let us differential the determinant $\det \mathcal{M}$ with respect to t . We have

$$\begin{aligned} \frac{\partial}{\partial t} \det \mathcal{M} &= \det \begin{pmatrix} \partial_t M_{00} & M_{01} & \cdots & M_{0,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_t M_{2n-1,0} & M_{2n-1,1} & \cdots & M_{N-1,N-1} \end{pmatrix} \\ &+ \det \begin{pmatrix} M_{00} & \partial_t M_{01} & \cdots & M_{0,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{2n-1,0} & \partial_t M_{2n-1,1} & \cdots & M_{N-1,N-1} \end{pmatrix} \\ &+ \cdots + \det \begin{pmatrix} M_{00} & M_{01} & \cdots & \partial_t M_{0,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N-1,0} & M_{N-1,1} & \cdots & \partial_t M_{N-1,N-1} \end{pmatrix}. \end{aligned}$$

Computing the individual determinants using the Laplace formula, we obtain

$$\frac{\partial}{\partial t} \det \mathcal{M} = \det \mathcal{M} \sum_{i,j=0}^{N-1} \partial_t M_{ij} \mu_{ji}. \quad (6.2)$$

As $r_j(x)$ are independent on t for $j < N - 2$, the derivative $\partial_t M_{ij}$ is given by

$$\partial_t M_{ij} = -\frac{1}{2} \left(\left\langle \frac{r_i}{t - \tilde{\tau}x}, r_j \right\rangle_1 + \left\langle r_i, \frac{r_j}{t - \tilde{\tau}y} \right\rangle_1 \right) = -\frac{1}{2} \left(\left\langle \frac{r_i}{t - \tilde{\tau}x}, r_j \right\rangle_1 - \left\langle \frac{r_j}{t - \tilde{\tau}x}, r_i \right\rangle_1 \right)$$

for $i, j < N - 2$. For either i or j equal to $N - 2$ or $N - 1$, we have

$$\begin{aligned} \partial_t M_{i,N-1} = & \delta_{N-2,i} \left(-\frac{1}{2} \left(\left\langle \frac{\pi_{N-2,1}}{t - \tilde{\tau}x}, \pi_{N-1,1} \right\rangle_1 - \left\langle \frac{\pi_{N-1,1}}{t - \tilde{\tau}x}, \pi_{N-2,1} \right\rangle_1 \right) \right. \\ & \left. + \langle \partial_t \pi_{N-2,1}, \pi_{N-1,1} \rangle_1 + \langle \pi_{N-2,1}, \partial_t \pi_{N-1,1} \rangle_1 \right). \end{aligned}$$

Note that by orthogonality, the last two terms in the above expression are zero as $\partial_t \pi_{N-1,1}$ is of degree $N - 2$ and $\partial_t \pi_{N-2,1}$ is of degree $N - 3$. Applying the same argument to $\partial_t M_{i,N-2}$, we obtain

$$\begin{aligned} \partial_t M_{i,N-1} = & -\delta_{N-2,i} \left(\left\langle \frac{\pi_{N-2,1}}{t - \tilde{\tau}x}, \pi_{N-1,1} \right\rangle_1 - \left\langle \frac{\pi_{N-1,1}}{t - \tilde{\tau}x}, \pi_{N-2,1} \right\rangle_1 \right) \\ \partial_t M_{i,N-2} = & -\frac{\delta_{N-1,i}}{2} \left(\left\langle \frac{\pi_{N-1,1}}{t - \tilde{\tau}x}, \pi_{N-2,1} \right\rangle_1 - \left\langle \frac{\pi_{N-2,1}}{t - \tilde{\tau}x}, \pi_{N-1,1} \right\rangle_1 \right). \end{aligned}$$

As \mathcal{M} is anti-symmetric, the derivatives $\partial_t M_{N-1,i}$ and $\partial_t M_{N-2,i}$ are given by $\partial_t M_{N-1,i} = -\partial_t M_{i,N-1}$ and $\partial_t M_{N-2,i} = -\partial_t M_{i,N-2}$. From these and (6.2), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \det \mathcal{M} = & \det \mathcal{M} \left(\sum_{i,j=0}^{N-3} \left\langle \frac{r_i}{t - \tilde{\tau}x}, r_j \right\rangle_1 \mu_{ij} \right. \\ & \left. + 2 \left(\left\langle \frac{\pi_{N-2,1}}{t - \tilde{\tau}x}, \pi_{N-1,1} \right\rangle_1 - \left\langle \frac{\pi_{N-1,1}}{t - \tilde{\tau}x}, \pi_{N-2,1} \right\rangle_1 \right) \mu_{N-2,N-1} \right), \end{aligned} \quad (6.3)$$

where we have used the anti-symmetry of \mathcal{M} and μ to obtain the last term. From the structure of the matrix μ in (5.14), we see that the last term in (6.3) can be written as

$$\begin{aligned} & 2 \left(\left\langle \frac{\pi_{N-2,1}}{t - \tilde{\tau}x}, \pi_{N-1,1} \right\rangle_1 - \left\langle \frac{\pi_{N-1,1}}{t - \tilde{\tau}x}, \pi_{N-2,1} \right\rangle_1 \right) \mu_{N-2,N-1} = \\ & \sum_{i=0}^{N-1} \sum_{j=N-2}^{N-1} \left\langle \frac{r_i}{t - \tilde{\tau}x}, r_j \right\rangle_1 \mu_{ij} + \sum_{i=N-2}^{N-1} \sum_{j=0}^{N-1} \left\langle \frac{r_i}{t - \tilde{\tau}x}, r_j \right\rangle_1 \mu_{ij} \end{aligned}$$

From this, (6.3) and the expression of the kernel in (5.11), we obtain (6.1). \square

Appendix: A proof of the j.p.d.f. formula using Zonal polynomials

We present here a simpler algebraic proof of Theorem 1 using Zonal polynomials. Zonal polynomials are introduced by James [20] and Hua [18] independently. They are polynomials with matrix argument that depend on an index p which is a partition of an integer k . The real Zonal polynomials $Z_p(X)$ take arguments in symmetric matrices and are homogenous polynomials in the eigenvalues of its matrix argument X . We shall not go into the details of their definitions, but only state the important properties of these polynomials that is relevant to our proof. Readers who are interested can refer to the excellent references of [25], [22] and [27].

Let p be a partition of an integer k and let $l(p)$ be the length of the partition. We will use $p \vdash k$ to indicate that p is a partition of k . Let X and Y be $N \times N$ symmetric matrices and x_i, y_i their eigenvalues. Given a partition $p = (p_1, \dots, p_{l(p)})$ of the integer k , we will order the parts p_i such that if $i < j$, then $p_i \geq p_j$. If we have 2 partitions p and $p' = (p'_1, \dots, p'_{l(p')})$, then we say that $p < p'$ if there exists an index j such that $p_i = p'_i$ for $i < j$ and $p_j < p'_j$. Let the monomial x^p be $x_1^{p_1} \dots x_{p_{l(p)}}^{p_{l(p)}}$, then we say that $x^{p'}$ is of a higher weight than x^p if $p' > p$. Then the Zonal polynomial $Z_p(X)$ is a homogenous polynomial of degree k in the eigenvalues x_j with the highest weight term being x^p . It has the following properties.

$$\begin{aligned} (\text{tr}(X))^k &= \sum_{p \vdash k} Z_p(X), \\ \int_{O(N)} e^{-My \text{tr}(XgYg^T)} dg &= \sum_{k=0}^{\infty} \frac{(My)^k}{k!} \sum_{p \vdash k} \frac{Z_p(X)Z_p(Y)}{Z_p(I_N)} \end{aligned} \quad (\text{A.1})$$

These properties can be found in the references [25], [22] and [27]. Another important property is the following generating function formula for the Zonal polynomials, which can be found in [22] and [27].

$$\prod_{i,j=1}^N (1 - 2\theta x_i y_j)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \sum_{p \vdash k} \frac{Z_p(X)Z_p(Y)}{d_p} \quad (\text{A.2})$$

for some constant d_p . In particular, if (k) is the partition of k with length 1, that is, $(k) = (k, 0, \dots, 0)$, then the constant $d_{(k)}$ is given by

$$d_{(k)} = \frac{1}{(2k-1)!!}.$$

For the rank 1 spiked model, let us consider the case where all but one y_j is zero and denote the non-zero eigenvalue by y . Then from the fact that the highest weight term in $Z_p(Y)$ is $y_1^{p_1} \dots y_{p_{l(p)}}^{p_{l(p)}}$, we see that the only non-zero $Z_p(Y)$ is $Z_{(k)}(Y)$, which by the first equation

in (A.1), is simply y^k . Therefore the formulae in (A.1) and (A.2) are greatly simplified in this case.

$$\begin{aligned} \int_{O(N)} e^{-My\text{tr}(XgYg^T)} dg &= \sum_{k=0}^{\infty} (My)^k \frac{Z_{(k)}(X)y^k}{k!Z_{(k)}(I_N)}, \\ \prod_{i=1}^N (1 - 2\theta x_i y)^{-\frac{1}{2}} &= \sum_{k=0}^{\infty} \theta^k \frac{(2k-1)!! Z_{(k)}(X)y^k}{k!} \end{aligned} \quad (\text{A.3})$$

By using the generating function formula, we see that $Z_{(k)}(I_N)$ is given by

$$Z_{(k)}(I_N) = \frac{(N/2 + k - 1)! 2^k}{(N/2 - 1)!(2k - 1)!!}. \quad (\text{A.4})$$

By taking $\theta = \frac{1}{2t}$ in the second equation of (A.3), we see that

$$\prod_{i=1}^N (t - x_i y)^{-\frac{1}{2}} = t^{-\frac{N}{2}} \sum_{k=0}^{\infty} (2t)^{-k} \frac{(2k-1)!! Z_{(k)}(X)y^k}{k!}.$$

We can now compute the integral

$$S(t) = \int_{\Gamma} e^{Mt} \prod_{i=1}^N (t - x_i y)^{-\frac{1}{2}} dt$$

by taking residue at ∞ , which is the t^{-1} coefficient in the following expansion

$$e^{Mt} \prod_{i=1}^N (t - x_i y)^{-\frac{1}{2}} = \sum_{k,j=0}^{\infty} \frac{M^j t^{-\frac{N}{2}+j-k} (2k-1)!! Z_{(k)}(X)y^k}{2^k j! k!}.$$

This coefficient is given by

$$\begin{aligned} S(t) &= M^{\frac{N}{2}-1} \sum_{k=0}^{\infty} \frac{Z_{(k)}(X)(2k-1)!! y^k M^k}{2^k (N/2 + k - 1)! k!} \\ &= \frac{M^{\frac{N}{2}-1}}{(N/2 - 1)!} \sum_{k=0}^{\infty} \frac{Z_{(k)}(X)y^k M^k}{Z_{(k)}(I_N)k!} = \frac{M^{\frac{N}{2}-1}}{(N/2 - 1)!} \int_{O(N)} e^{-My\text{tr}(XgYg^T)} dg. \end{aligned}$$

This proves Theorem 1. There also exist complex and quaternionic Zonal polynomials $C_p(X)$ and $Q_p(X)$ which satisfy the followings instead.

$$\begin{aligned} \int_{U(N)} e^{-My\text{tr}(XgYg^\dagger)} g^\dagger dg &= \sum_{k=0}^{\infty} \frac{(My)^k}{k!} \sum_{p \vdash k} \frac{C_p(X)C_p(Y)}{C_p(I_N)}, \\ \int_{Sp(N)} e^{-My\text{Re}(\text{tr}(XgYg^{-1}))} g^{-1} dg &= \sum_{k=0}^{\infty} \frac{(My)^k}{k!} \sum_{p \vdash k} \frac{Q_p(X)Q_p(Y)}{Q_p(I_N)}. \end{aligned}$$

Their generating functions are given by

$$\prod_{i,j=1}^N (1 - 2\theta x_i y_j)^{-1} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \sum_{p \vdash k} \frac{C_p(X)C_p(Y)}{c_p},$$

$$\prod_{i,j=1}^N (1 - 2\theta x_i y_j)^{-2} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \sum_{p \vdash k} \frac{Q_p(X)Q_p(Y)}{q_p}$$

where $c_{(k)}$ and $q_{(k)}$ are

$$c_{(k)} = \frac{1}{2^k k!}, \quad q_{(k)} = \frac{1}{(k+1)! 2^k}$$

Then by following the same argument as in the real case, we can write down the following integral formulae for rank one perturbations of the complex and quaternionic cases.

$$\int_{U(N)} e^{-My \text{tr}(XgYg^\dagger)} g^\dagger dg = \frac{(N-1)!}{M^{N-1}} \int_{\Gamma} e^{Mt} \prod_{i=1}^N (t - x_i y)^{-1} dt,$$

$$\int_{Sp(N)} e^{-My \text{Re}(\text{tr}(XgYg^{-1}))} g^{-1} dg = \frac{(2N-1)!}{M^{2N-1}} \int_{\Gamma} e^{Mt} \prod_{i=1}^N (t - x_i y)^{-2} dt. \quad (\text{A.5})$$

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